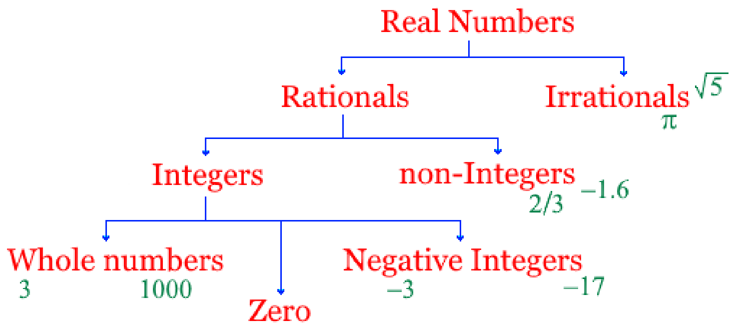
Math to Ponder While the Wave Passes

The last column dealt with irrational numbers, and there were two problems for you to ponder. The first was to prove that $\sqrt{5}$ is irrational, and the second was to prove that the line given by $y = \sqrt{2}\,x$ would never touch any lattice point except $(0, 0)$, even though it has infinite length, and the lattice points extend infinitely in all directions. Solutions are at the end.

Grab another mug of coffee, and let's continue with one of our themes, namely, that mathematics is the provider of all the types of numbers that civilizations need, along with their logical structure. A short summary seems useful here. We have seen that whole numbers are founded in deep prehistory, but children still use them to count Legos. These are an example of something both simple and profound. Next, several civilizations developed rational numbers, in order to meet their needs in astronomy, commerce, and land measurements, for instance. Irrational numbers came into comprehension through Greek geometry. Their arithmetic structure was established by Eudoxus, right along with that of rational numbers. Negative numbers (rational and irrational), were fully accepted after Newton gave the simple rules for handling them.

And so it stood, with an emerging understanding that the nature of numbers was to be found in the dichotomy between rationals and irrationals. Mathematicians began thinking of the ensemble of rationals and irrationals (regardless of their sign) as the set of **real** numbers, perhaps to insulate themselves from the unwelcome discovery in the 1500's of some quantities called imaginary numbers. More about those "phantasms" in a later column.

It took a short time after Newton's work for mathematicians to put into order the numerical landscape before them. Rational numbers were just a special type of real number, and integers were just a special type of rational number—which implied that integers were just an extra-special type of real number. What was coming into focus was what is called a taxonomy, a branching relationship similar to that of species, genus, family, etc.. Biologists say that the family named "Canidae" is composed of several genera, one of which is named "Vulpes," which itself contains a species called "red foxes." Mathematicians likewise say that the set of real numbers is composed of rationals and irrationals, and that the set of rationals is itself composed of the sets of integers and non-integers. Thus we find the "tree" relationship below (insignificant variations are seen in the Web).



The successively more inclusive sets running up the left side are of interest to us. We see them as numbers capable of solving increasingly more complex equations. Thus, $2x + 5 = 9$ can be solved by lowly whole numbers, but they can't solve $2x + 5 = 1$. However, with the set of integers we *can* solve $2x + 5 = 1$, since they include the set of negative integers. We may say that the set of integers is richer than the set of whole numbers. Unfortunately, integers are not "rich" enough to solve $2x + 5 = 8$. But the more inclusive set of rationals can. Rationals in turn show their insufficiency in that they cannot solve $x^2 + 6x - 3 = 0$, so we need the even richer set of reals (yes indeed, this is because the solutions are irrational numbers, brought to you by the world-famous quadratic formula).

It turns out that real numbers are supremely rich. This set is capable of supplying solutions to much more complicated equations, such as $-\frac{33}{2}x^4 + 55x^3 + 253x^2 - 385x - \frac{1925}{2} = 0$, a polynomial equation of fourth degree, and $\cos(4y) + \sin(2y) = 1$, a trigonometric equation, and $20 = \dfrac{10(850)}{10 + 840e^{-3.73t}}$, a Verhulst equation from weeks ago in this series.

Considering how tricky it was to nail down the definition of "whole number" (discussed in a previous column with regard to "three"), it seems out of reach to try to characterize a real number. But math must do that. An engineer, for example, needs to know exactly how to handle the numbers involved in any of that field's formulas. If this seems far from common life, it's because such critical details have been hammered out long ago. Misuse of numbers could lead to outrageous results. For instance, suppose a bank for some reason decided to consider money as a complex number, i.e., a number involving the imaginary quantities mentioned above. Then things could be set up so that two sizable deposits made on different days would give you a zero balance! So, finance is complicated, but not "complex" in our way of thinking.

Now, let's see how mathematics distinguishes real numbers from every other idea conceived by the human mind. You already know some of the building blocks: the associative law, and the commutative law. The process of defining real numbers starts with taking a set of unspecified things, called $\mathbb{R}$, from which we take any three and label them a, b, and c. We will create two operations to be done on these things, suggestively called "addition" and "multiplication," and still more suggestively, we use the symbols "+" and "×". Don't think that this tells you anything about the operations. Lewis Carroll—true name Charles Dodgson—once poked fun at them by renaming them "ambition" and "uglification" (and he piled it on by calling the other two operations "distraction" and "derision").

By the way, Carroll wasn't just the acclaimed author of the Alice in Wonderland series. He was a first-rate logician. There is a story that Queen Victoria was so delighted by the children's stories that she insisted Carroll send her the very next thing he published. It had to have been *Euclid and his Modern Rivals* (1879), a study of the logic in Euclid's great mathematical books! If the story is true, we can be sure that the Queen was not amused.

Anyway, a, b, and c are just objects of interest, and + and × are just actions to perform with the objects. Next, we create 11 rules, or **axioms**, which these objects must obey. Some of them seem too obvious to worry about, while others are quite profound:

(1) $a + b$ and $a \times b$ are both unique members of $\mathbb{R}$ (that is, + and × give unique answers).

(2) $a + b = b + a$ and $a \times b = b \times a$ (this is the commutative law).

(3) $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$ (this is the associative law).

(4) $a \times (b + c) = a \times b + a \times c$ (this is the distributive law).

(5) There is a unique member of $\mathbb{R}$, call it 1, such that for any a, $1 \times a = a \times 1 = a$ (this says that 1 is the multiplicative identity for $\mathbb{R}$).

(6) There is a unique member of $\mathbb{R}$, call it 0, and different from 1, such that for any a, $0 + a = a + 0 = a$ (this says that 0 is the additive identity for $\mathbb{R}$).

(7) For every a in $\mathbb{R}$, there is a unique member in $\mathbb{R}$, written $-a$, such that $a + (-a) = (-a) + a = 0$ (every a has an additive inverse, $-a$).

(8) For every a in $\mathbb{R}$ except 0, there is a unique member in $\mathbb{R}$, written a', such that $a \times (a') = (a') \times a = 1$ (every a except 0 has a multiplicative inverse, a').

(9) There exists a subset of $\mathbb{R}$, called $\mathbb{R}^+$, such that for all a except 0, exactly one of a and $-a$ is in $\mathbb{R}^+$. Also, if a and b are in $\mathbb{R}^+$, then $a + b$ and $a \times b$ will be in $\mathbb{R}^+$ (this distinguishes positive from negative).

(10) The principle of mathematical induction. This allows us to prove statements that have infinitely many cases.

(11) The completeness axiom. Roughly, this allows us to distinguish and locate all irrationals among the rationals, so the number line is properly filled in.

The last two axioms are a bit too abstract to state precisely here. The first ten axioms perfectly describe the rational numbers. That is, *any* set of things that obey those axioms will act *precisely* as the set of rationals do, so we might as well call any such set "rational numbers." Now, throwing in the completeness axiom is like adding yeast to what is merely dough. Completeness is, for example, the axiom upon which the edifice of calculus rests, completely resolving the qualms of Bishop Berkeley (see a previous column). We are assured that any set of things that obey these 11 axioms will be essentially indistinguishable from the reals, so we may call that set "real numbers."

Notice that nothing is used before it is identified and defined. This is a matter of logic. Thus, a' in axiom (8) is apparently nothing more than the unit fraction $\frac{1}{a}$ of Egyptian fame (we know that $a \times \frac{1}{a} = 1$). However, division has not been defined yet, so the expression $\frac{1}{a}$ is a big question mark. But once division is defined, $\frac{1}{a}$ will take the place of a'. And we have already seen why axiom (8) explicitly excludes 0. It is also interesting that axiom (6) explicitly says that $0 \neq 1$. This seems ridiculous, but if it weren't there, a nasty lawyer may see it as a loophole and insist that $0 = 1$. This would cause the entire house of axioms to crash due to contradictions. Axiom (9) seems like a weird way to define positive numbers, but remember that we can only use expressions that were defined previously. And it says that if you're not a positive or 0, then you will be a negative. It also says that once you are in the positive subset $\mathbb{R}^+$, you can't get out of it by addition or multiplication. Fair enough!

Axiom (9) doesn't state what the status of 0 is. But we can prove that 0 is not in the positive subset $\mathbb{R}^+$. Suppose it is in $\mathbb{R}^+$. Then, by axiom (9), exactly one of 0 and –0 is in $\mathbb{R}^+$, so –0 cannot be in $\mathbb{R}^+$. But we can actually show that $-0 = 0$. Thus, –0 is also in $\mathbb{R}^+$. This contradicts axiom (9), and we have a *reductio ad absurdum* situation. Thus, 0 is not positive, and it won't be negative, either. It's all alone, the poor thing. A very important fact now emerges. We have proved conclusively that "One is the loneliest number..." is an undeniably false line in the rock-and-roll song by Three Dog Night. Ah, well, romantics...

There is also a building-block property. I mentioned that by just inserting axiom (11), the things described jumped from rationals to reals. We can reverse the process, too. If, from the first ten axioms we delete (8), then the remaining ones define the set of integers perfectly. (Suddenly, there are gaps between numbers, and most don't have reciprocals.) If we get really brave and remove (6), (7), and (9), the remaining axioms define the set of whole numbers perfectly. It may not be a surprise that mathematicians have gone bonkers over this building-block idea, deleting and inserting other axioms, and thereby creating sets with very strange properties. One of these, in particular, is called a "group," and its structure has become essential in—of all things—quantum theory!

So, our 11 axioms not only define real numbers, but are also an interlocking set of instructions for how to work with them. They are never contradictory, and each one is necessary, and they are all that is needed until one wanders far into abstruse regions of mathematics. This level of understanding was not achieved until the latter half of the nineteenth century. The axiomatic structure of quantities is one of the pinnacles of modern mathematics.

Is there a fly in the ointment? No, but there are those pesky phantasms called imaginary numbers... .

From the last column, we had to prove that $\sqrt{5}$ is irrational. We try the same technique as for $\sqrt{2}$. Assume that $\sqrt{5}$ is rational, and let $\frac{a}{b} = \sqrt{5}$ be the unique reduced fraction. Then $a^2 = 5b^2$, but now we don't see evidence of evenness. Still, a^2 is displaying a factor of 5; hence, a must have a factor of 5. We can write $a = 5m$. Then $a^2 = 25m^2 = 5b^2$, so $5m^2 = b^2$. This is looking familiar! Then b must have a factor of 5, and therefore, $\frac{a}{b}$ was not reduced. A *reductio ad absurdum* case again, proving that $\sqrt{5}$ is irrational.

Next, suppose that the very strange line $y = \sqrt{2}\,x$ is not so strange, because it does intersect some lattice point (a, b) (it doesn't matter if a and b are in the trillions). Substituting, get $b = \sqrt{2}\,a$, which implies that $\sqrt{2} = \frac{b}{a}$. We know this is impossible, so by *reductio* again, the given line *never* crosses any lattice points (except the origin). Impossible to visualize, but true. It's even more inconceivable than that. The line doesn't cross through any point with *any* pair of rational coordinates whatsoever (origin excluded)! This is easy to see: suppose the intersection point is $(147.33, 208.35)$. A high-res screen certainly shows this to be so. But then, $\sqrt{2} = \frac{208.35}{147.33} = \frac{20835}{14733}$, false, although very close. So the line misses every single point with rational coordinates, except the trivial origin. Well, what's left for the line to cross? The answer is the unimaginable infinity of points with one or more irrational coordinates. For example, it crosses through $\left(\sqrt{2}, 2\right)$. One wonders for a moment if such a line really exists. It does, but not in the physical universe!

Luis F. Moreno, professor emeritus, Mathematics Department.