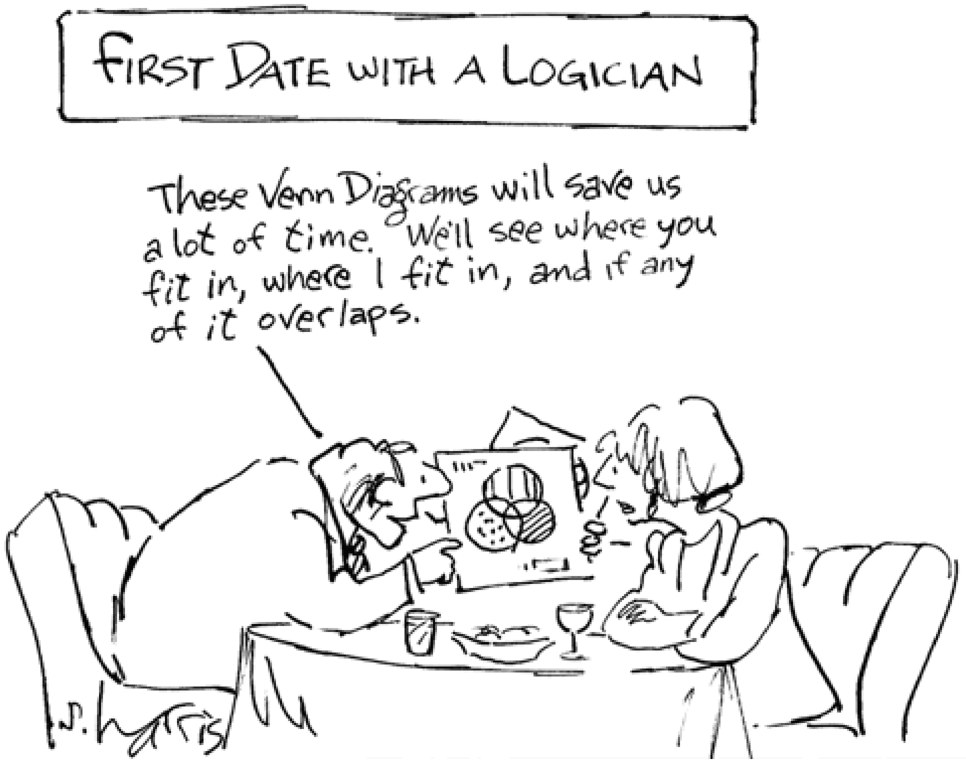

Math to Ponder While the Wave Passes

The last column left you with a plausible problem for a chef. A recipe calls for $\frac{1}{12}$ cup of water. Can that much water be precisely measured out with these measuring cups: $\frac{1}{2}$ cup, $\frac{1}{3}$ cup, $\frac{1}{4}$ cup, and $\frac{1}{8}$ cup? Yes, it can. First, fill the $\frac{1}{3}$ cup and pour it to fill the $\frac{1}{4}$ cup. This leaves $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ cup in the larger cup.

Today, we go further into our definition of mathematics as the study that begins by providing all the numbers that any civilization needs, along with the requisite logical structure. We consider rational numbers, or fractions. The problem above takes us back to ancient Egypt, where it seems that the only fractions around were single pieces of a whole. That is, Egyptians were comfortable with the fractions of a cup listed above, along with $\frac{1}{5}$, $\frac{1}{6}$, etc., and possibly $\frac{2}{3}$, but not with things like $\frac{3}{5}$, or $\frac{7}{8}$, or $\frac{12}{8}$! The Ahmes Papyrus of around 1650 BC contains arithmetic that is probably 2000 years older [*A History of Mathematics*, 2nd ed., by C. Boyer and U. Merzbach, Wiley, 1991, pp. 11-14]. Needless to say, having only "unit fractions," with numerator strictly 1, would make basic arithmetic with fractions miserable. Perhaps this was purposeful, for the experts in mathematics were closely allied with the scribes, and the entire caste was kept under the control of the Pharaohs. Thus, math was part of the esoteric knowledge of the ages. I know what you're thinking: "it still is..." Well, not so much, anyway. But it is the universal language of the 21st century.

At any rate, something as simple as $\frac{3}{5}$ would be thought of as the sum $\frac{1}{3} + \frac{1}{5} + \frac{1}{15}$ by Ahmes. I suspect that long tables of sums like these were stored in the scribes' homes, similar to the tables of trigonometric functions *we* used to keep. Imagine teaching children to add $\frac{3}{5} + \frac{7}{8}$ by converting both to unit fractions! (I pray it won't be bundled into the next Common Core curriculum.) This brings up a classically mathematical question: can any fraction $\frac{p}{q}$ be written as the sum of unit fractions? Yes, and a way it can be done is described by an algorithm — a very old word derived from the name of Arab mathematician Al-Khwarizmi (c. 780 - c. 850). We're not going to merely "walk like an Egyptian," as Steve Martin says, but we're going to do arithmetic like an Egyptian.

An **algorithm** is a set of clearly defined steps that eventually produces an output. Here, then, is the algorithm for decomposing $\frac{p}{q}$ Egyptian style: (1) Find the largest unit fraction below $\frac{p}{q}$ (less than it); (2) Subtract the unit fraction just found from $\frac{p}{q}$; (3) If the difference from step (2) is not a unit fraction, apply step (1) to the difference, and keep doing (1) and (2) until the difference is a unit fraction; (4) When the difference is a unit fraction, stop. The output is the unit fractions discovered in the several returns to step (1) along with the final one from (4).

Using this algorithm with $\frac{7}{8}$ produces these steps:

(1) $\frac{1}{2}$ is the largest unit fraction below $\frac{7}{8}$

(2) $\frac{7}{8} - \frac{1}{2} = \frac{3}{8}$, not a unit fraction

(3) $\frac{1}{3}$ is the largest unit fraction below $\frac{3}{8}$

(4) $\frac{3}{8} - \frac{1}{3} = \frac{1}{24}$

Therefore, $\frac{7}{8} = \frac{1}{2} + \frac{1}{3} + \frac{1}{24}$. For a pastime with an espresso at your side, try finding $\frac{8}{17}$ Egyptian style. I'll post the answer in the next column.

Well, we see that an algorithm has such clearly defined steps that even a computer could follow it. Precisely. Computer programs are algorithms, often very complex algorithms. The apps that you download to your phone are algorithms. The probes on and around Mars operate with algorithms, and these better be flexible and smart, for there is no hacker out there!

Two interesting questions arise about this algorithm (and indeed, any algorithm). First, is the output unique? The answer is that no, there may be other unit fractions that add up to $\frac{p}{q}$. For example our algorithm gives $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$, surely a nicer answer than the one Ahmes wrote up above. Second, will it stop? Step (4) doesn't just say, "Stop." How do we know that the difference will eventually be a unit fraction? In fact, it fortunately will be. The proof of that takes us too far from the purposes of this column, but nevertheless, try it. It's hard to believe that such an innocent question as "Will it stop?" opened up a Grand Canyon-sized problem in math and logic. It was solved by Alan Turing in 1937 in his seminal paper, *On Computable Numbers With an Application to the Entscheidungsproblem*. (The German word means "decision problem.") His astonishing answer was that there will always exist algorithms whose stopping will be forever undecidable. This isn't figurative language. No computer, however complex, will ever be able to determine in all cases if an algorithm will eventually halt, or not. This represents a definite cap on what is knowable, even for advanced aliens!

Going forward, the Babylonian civilization progressed by using a base 60 numeration system. Sixty has many factors, so quite a few fractions could be easily expressed. For instance, since $\frac{2}{3} = \frac{40}{60}$, scribes of the time would first write a symbol alerting the reader that what followed was sixtieths, and then write their digits for "40". Wonderful! Likewise, $\frac{7}{10} = \frac{42}{60}$. The first fraction to give problems was $\frac{1}{7}$. Using the our algorithm but with sixtieths, they would find $\frac{1}{7} - \frac{8}{60} = \frac{1}{105}$, so the first fraction digit would be "8". Since $\frac{1}{105} < \frac{1}{60}$, they would go for the next place value fraction, $\frac{1}{60^2} = \frac{1}{3600}$. It turns out that $\frac{1}{105} - \frac{34}{3600} = \frac{1}{12\,600}$ but $\frac{1}{105} - \frac{35}{3600} = -\frac{1}{5040}$, so after the "8" they would write a symbol for a three-thousand-six-hundredth, followed by "34". They understood that the algorithm would not halt (aha!), so $\frac{1}{7} \approx \frac{8}{60} + \frac{34}{60^2}$ with an error so small that it could be forgotten. Base 60 was so useful that Greek mathematicians were still using it for numerical calculations a millennium later.

It took the ancient Greek mathematicians to forge the set rational numbers into a logical system, mainly by the hand of Eudoxus (c. 408 - c. 355 BC, a professor in Plato's Academy). They still did not write fractions in the way we do, say $\frac{7}{8}$, but instead they thought in terms of ratios of lengths, as in 7 stadia to 8 stadia, commonly written today as $7 : 8$. To them, seven and eight were **commensurate** numbers, since a multiple of seven would equal some multiple of eight. The fifth book of Euclid's massive compendium of math cemented the arithmetic of ratios, around 300 BC. There were 18 definitions and 25 propositions, all written longhand, for there was no algebraic symbolism back then. It was tough reading. Here is a typical definition which is very fundamental to all real numbers, not just fractions: "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another." This says that $47 : 5$ is a ratio since 5×10 does exceed 47. Another definition from that book: "Let magnitudes which have the same ratio be called proportional." We write this today as $\frac{a}{b} = \frac{c}{d}$. Other propositions deal with inequalities, such as "If $x : z < y : z$, then $x < y$...", written with our algebra. This statement appears in every algebra book published today, but with an addendum if z be negative, which, if you remember, was not recognized in those times. So, as mentioned in a previous column, mathematics is permanent and cumulative. [Quotes are from Book V of Euclid's *Elements*, at https://mathcs.clarku.edu/~djoyce/java/elements/bookV/bookV.html]

The fraction bar, or vinculum, as in $\frac{7}{8}$, was brought to Western Europe by Leonardo Fibonacci in his *Liber Abaci* (1202), the same book in which he introduced the ten Hindu-Arabic numerical digits. All of this had been used by some Arab mathematicians long before. The solidus, or slash, as in 7/8, was suggested centuries later by Victorian logician Augustus DeMorgan [Boyer and Merzbach, pg. 254]. I imagine this relieved the stress of typesetting regular fractions within running lines of text, while keeping everything in line. Even today it is irritable to write "$\frac{317}{365}$" with a word processor. I also imagine that the obelus, the symbol in $7 \div 8$, is the result of merging the colon and vinculum together. It's worth noting that the vinculum is super handy as a grouping symbol, replacing four parentheses in expressions such as $\frac{4-7}{13+55}$.

When the graphs of analytic geometry were introduced to mathematics by René Descartes in 1637, the slope of a line was distinguished as the fraction $\frac{rise}{run}$. Thus, a slope of $\frac{2}{5}$ for a roof meant that for every two feet of rise, there must be five feet of run. All was good until someone asked for the slope of a vertical wall. The run was zero, regardless of the rise. Suppose the rise were seven. This produced the slope $\frac{7}{0}$, which was always recognized as an impossibility. I should explain why this is so. Remember that if $\frac{a}{b} = c$, then we can check the division by seeing that $a = bc$. But $\frac{7}{0} = c$ never checks: we get $7 = 0c = 0$, regardless of the value of c. Thus, division by zero is impossible. Alas, things got really murky when Isaac Newton and Gottfried Leibniz discovered calculus in the 1670's. Their analysis led them to consider the ratio $\frac{0}{0}$. Again assuming that $\frac{0}{0} = c$, we would then check that $0 = 0c = 0$. Excellent, so $\frac{0}{0} = 0$. But not so fast, the check works for *any* number c. Thus, $\frac{0}{0}$ could equal 416.3. Shades of Humpty Dumpty in the previous column, $\frac{0}{0}$ could equal any number we desire! Hence, $\frac{0}{0}$ was also banished as impossible, but for a different reason than $\frac{7}{0}$.

What was worse was that under Newton's and Leibniz's careful study, what looked like $\frac{0}{0}$ turned out to be the speed of an object at any given instant of time! Immediately, math/physics (they were one department at the time) became polarized against the philosophers. The believers in calculus agreed that $\frac{0}{0}$ wasn't exactly that, but the "ratio of vanishing quantities," which invariably produced the correct velocities. The logicians, represented by Bishop Berkeley, would have absolutely none of that rubbish. Their position was that a ratio was either $\frac{0}{0}$, or it wasn't, with no grey area in between. So, if correct velocities were being evaluated, and geometry was correct, then calculus must be a bastardization of Euclid! Fortunately, the wars waged between these adversaries took place only within the walls of academia. It was eventually settled, after about 150 years, and the reconciliation was permanent, in contrast to treaties between countries.

And so it is that decimals, percents, and other forms of fractions have their firm foundation within mathematics, readily available whenever civilization needs them. In the next column, I will talk about the problem of incommensurable quantities.

Finally, another curious problem using fractions. To keep things nice, let *a*, *b*, *c*, and *d* be whole numbers. Form two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, such that $\frac{a}{b} < \frac{c}{d}$. Adding fractions is a bit tricky unless you have practiced it. Using our very efficient algebra, the sum $\frac{a}{b} + \frac{c}{d}$ is done by raising the terms of each fraction, $\frac{ad}{bd} + \frac{bc}{bd}$, which does *not* change the value of each fraction, but *does* produces a common denominator, *bd*. We can now finish the job by adding the numerators, to get $\frac{ad+bc}{bd}$. Next, we know that the average of two numbers is always midway between them; we find it by adding the numbers and dividing by two. Thus, the average of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{1}{2}\left(\frac{ad+bc}{bd}\right)$. And we know that $\frac{a}{b} < \frac{1}{2}\left(\frac{ad+bc}{bd}\right) < \frac{c}{d}$. Now, kids who don't do their homework will think that $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$, that is, just add straight across. This result is not the sum, but it has interesting properties. One is that this is the way a baseball player's batting average is updated as the season passes. For instance, if a player has 8 hits and 22 times at bat, his current average is $\frac{8}{22} \approx 0.364$. If on the next game he gets two hits with four times at bat, his new average is $\frac{8+2}{22+4} \approx 0.385$. Another property is that like the average, $\frac{a+c}{b+d}$ is between the two original fractions. Your job is to prove that this is true, that is, that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Hint: try it with particular whole numbers for *a*, *b*, *c*, and *d*, to get the feel of it.

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