Math to Ponder While the Wave Passes.

This is the fourth in a series of columns on mathematics not common in classrooms, but designed to stretch your cup of java.

We left the last article with Archimedes's discovery of the first law of exponents that said $10^a \times 10^b = 10^{a+b}$ (in modern notation). During his time, two other types of numbers were recognized: rational numbers (thought of as ratios), and irrational numbers. In a logical development, zero and negative whole numbers should come next. However, quantities less than nothing were considered fictitious. After all, how could one have *less* than no cubits of string? And zero? Zero was tied up with the problem of the existence of "nothing," which was philosophically as hard to swallow as a cactus.

Aristotle, the founder of logic, would have nothing to do with the "empty set," the collection which contains nothing. This is still obtuse to the mind, for if one has nothing, how does one make a collection of it? Nevertheless, modern set theory refers to the empty set often. Notation is helpful here. The set containing the solution to the equation $5x + 10 = 30$, the Empire State Building, and the name of the most famous novel by Herman Melville looks like this: $\{4, ESB, Moby\,Dick\}$. Here *ESB* is a symbol to represent the said building (I wasn't able to rip the actual building off its foundations by myself). The braces tell us that this is a set, and we see that it has three members. Aristotle would be content, although a bit perplexed about what the three items had to do with each other. Now then, the empty set is written { }. The braces indicate the existence of a set, but it's what's inside the braces that is significant! Aristotle would say, "My dear fellow, this is nonsense. There is nothing there." But that was the whole point.

Throughout ancient and medieval mathematics, negative quantities were not accepted as numbers. So, when an equation had negative solutions, these would be rejected, and the equation would be called "unsolvable." But Hindu accountants had introduced negative numbers to represent debts, while positive numbers acted as assets. This was clever, because they did not have to use red ink and black ink for those purposes, as European accountants did. The mathematician Brahmagupta (620's AD) seems to have been the first to use negative numbers in a mathematical sense. In other words, he obtained the general mathematical principles for working with negative numbers. Thus, having a debt of $100 and then gaining $60 needn't have the accountant write a red "40" in the ledger. Brahmagupta understood that $-100+60 = 60+(-100) = -40$, and this was the nature of addition with negative numbers. Nevertheless, negative numbers were accepted very slowly. Here is Bhāskara, 400 years after Brahmagupta, writing about 50 and −5, two solutions to an equation, "the second value is in this case not to be taken, for it is inadequate; people do not approve of negative solutions." [*Mathematical Thought from Ancient to Modern Times*, by Morris Kline, Oxford Univ. Press, 1972, pg. 185.] So much for the wisdom of public opinion!

Isaac Newton published his *Arithmetica Universalis* in 1707, which finally established algebra on an equal footing with geometry. In it, he explained negative integers as numbers to be placed to the left of zero in size order, as mirror images to positive integers, which extend to the right. Thus, they take their place in a completed number line. This was elegant, beautiful, and pregnant with meaning. One could actually see the process behind $60+(-100) = -40$. And he further enunciated such properties as negative × negative = positive, etc., with clear mathematical reasoning. Newton's clout was so great that ever since then, the entire range of integers, written in set notation as $\{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$, has found a secure home in math and the sciences. Once again, mathematics provides the numbers that civilizations need.

Incidentally, I think the keyboards we use today still feel the hangover from the ages of distrust of negatives. Notice that the "+" symbol is readily available, but there is no "−" symbol to be seen. The closest is the hyphen, "-". But it has its own grammatical use, and looks too stubby when we write -40. How does one distinguish between the phone number 778 - 5000 and the subtraction 778 - 5000? "By context," is the stock answer. I vote that the grave accent (the backwards apostrophe) be removed form the keyboard on the grounds that it is practically never used, and that it be replaced by the proper "minus" symbol.

Last, and seemingly least, we have not forgotten zero. Let's match several simple sets with the number of members they contain:

$$\begin{aligned} \{a\} &\leftrightarrow 1 \\ \{a,b\} &\leftrightarrow 2 \\ \{a,b,c\} &\leftrightarrow 3 \\ \{a,b,c,d\} &\leftrightarrow 4 \end{aligned}$$

This one-to-one correspondence (aha, the hyphen!) is as simple and clear as can be. It takes us back to prehisoric counting. But it is profound and powerful. In fact, this idea may be used to *define* whole numbers. Have you ever tried to define "three"? Dictionaries end up with definitions that saw the branch which holds them up. For instance, "Three is the integer following two." Well, then, what is "two"? You see the slippery slope towards zero. And the lexicographers are in deeper trouble still, for we must logically ask, "What is an integer?" Their reply, "A member of the set $\{0,1,2,3,\cdots\}$ or $\{-1,-2,-3,\cdots\}$." To our dismay, we see the "3" object. Perfectly circular definitions these are, but perhaps acceptable since the words "three" and "integer" are separated by hundreds of pages.

Circular definitions are completely unacceptable in mathematics. Roughly, the definition of "three" is: the name of the collection of all sets with the one-to-one correspondence to $\{a,b,c\}$. Our other set, $\{4, ESB, Moby\,Dick\}$, would do just as well in this definition. Abstract, yes, but not circular. On the contrary, the definition captures the sense of threeness. And finally, since $\{a\} \leftrightarrow 1$, what matches { }? It makes sense that $\{\ \} \leftrightarrow 0$. It turns out that all of arithmetic can be built up from these and similar primitive notions, and nothing is circular or contradictory. Never can it be said of numbers what egotistical Humpty Dumpty once said about a word, “it means just what I choose it to mean—neither more nor less.”

I leave you with an arithmetic problem that points us to the next column in this series. You have a standard set of measuring cups, consisting of 1 cup, $\frac{1}{2}$ cup, $\frac{1}{3}$ cup, $\frac{1}{4}$ cup, and $\frac{1}{8}$ cup. A recipe calls for $\frac{1}{12}$ cup of water. Can that much water be precisely measured out with the standard set of cups?