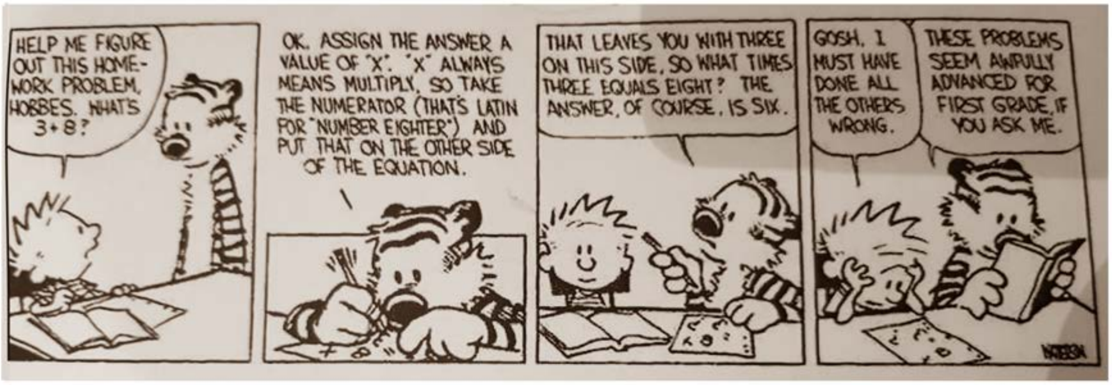


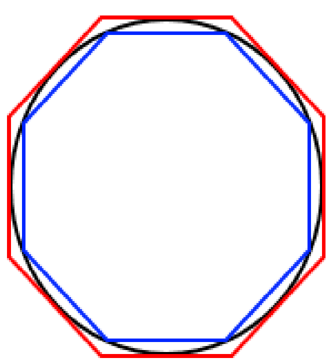
Math to Ponder While the Wave Passes

The answer to the previous column's question about the infinitely crinkled Koch curve is at the end.

This time, let's look at some of mankind's scuffles with the infinite. Here and there in these columns, we've broached the subject, but rushed past it. In the column about whole numbers and counting, we saw that something so numerous that it couldn't be counted made it tantamount to infinity. In the last column, I talked about astonishing curves that live in between dimensions, one of which is the Koch curve. I described it as "infinitely" crinkled. What does this word entail?

The ancient Greek mathematicians carefully steered away from infinity. Euclid's great work of geometry and arithmetic, the *Elements* (300 BC), never uses an infinitely long line. For instance, he skirts the problem when talking about parallel lines: "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction" (Book I, definition 23). So, "produced indefinitely" means "regardless of how long," which itself *almost* means "infinitely long." But the terms are not synonymous. Euclid is not allowing an actual infinity of length, only the potential infinity of length. It's a subtle distinction, but crucial. A **potential infinity** of length or quantity describes any process that is not bounded in some aspect of length or quantity. Thus, producing a line indefinitely meant that at any given step, the line is always a finite segment. But one must never bound the process, for obviously, if one placed a bound on the two lines' length, then they may or may not turn out to be parallel. So, Euclid was as clever as a fox in excluding the problem of actual infinity.

The greatest mathematician and physicist of the ancient world, Archimedes, dealt with a potential infinity in quantity and length when he made his groundbreaking analysis of π. He describes a process of inscribing and circumscribing regular polygons around a circle. Below is a circle with a blue inscribed octagon and a red circumscribed one. Letting the circle's diameter be 1, its circumference is π, and a high school student who does his/her geometry homework faithfully will be able to calculate the perimeter of the two octagons. We won't do it,

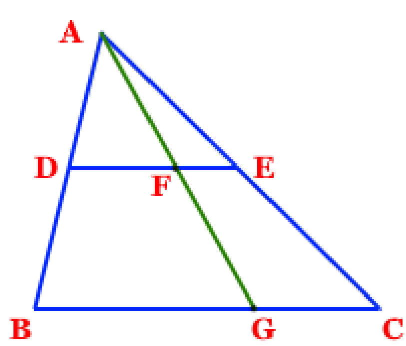


but it is pretty convincing that if p_8 is the blue perimeter and P_8 is the red perimeter, then $p_8 < \pi < P_8$. Archimedes observes that as the number of sides n of the polygons increases, the inside perimeters increase towards π, and the outside perimeters decrease towards π. (It's amazing that in his day, Archimedes was able to handle $n = 96$ sides and discover the first rigorous approximation of π: $3.1408 < \pi < 3.1429$.)

Infinity rises again. The number n is always finite, but increases without bound. It exemplifies potential infinity. It makes no sense to ask, "How large is it, finally?" Along with the increase in n, the polygons begin to look more and more like the circle sandwiched in between. Can we claim therefore that the circle is a polygon with infinitely many sides? It's tempting, except that every side would be reduced to zero length, which is not a side of anything at all. The sequence of polygon's perimeters is potentially infinite, but at the same time, it is not infinitely long. Instead, it describes a process that approaches π.

This substitution of potential infinity for actual infinity was Aristotle's way of trying to tame the paradoxes that surrounded actual infinity. It wouldn't be an exaggeration to say that Zeno's paradoxes from about 450 BC had scared the early schools of mathematicians witless. As I've mentioned in a past column, the only thing that remained on solid ground was deductive proof. Aristotle began to see that Zeno had intermixed actual infinity with potential infinity. For example, the paradox of the flying arrow states that an arrow certainly possesses a certain speed in the course of its flight. And it must also traverse every point of its path. But at every such point, the arrow cannot be moving, since a point has no length. Hence, no arrow can reach its destination. Part of the problem is that Zeno is mixing a physical problem with mathematical ideals. But this hardly resolves the paradox. Aristotle says that the problem lies in attempting to use an actual infinity of points instead of a potential infinity. The actual infinity cannot be accomplished, but the process involves traversing a finite path, which is easily accomplished since it must be considered a potential infinity. There is much similarity with Archimedes's polygons which have finitely many vertices on the circle at every step, and the circle itself, which has an actual infinity of points already on it. (A difference is the physical entity of speed, which doesn't exist in Archimedes's construction, being purely mathematical. The careful description of speed would have led old Zeno to mark his paradox as resolved, but that would take the discovery of calculus, millennia in the future.)

At any rate, Euclid and Archimedes bypassed these philosophical problems. But the problem of infinity persisted. Medieval theologians were comfortable with infinity as an attribute of God, for if it were not, then even fallible man could imagine a being better than this god. But again, these were philosophical considerations, not related to quantity or length. Galileo (1564-1642), on the other hand, recreated a monstrous paradox of actual infinity with a simple pair of similar triangles. He asserts along with everyone that a line segment has infinitely many points. Now, consider triangles ABC and ADE below. They are similar, and so constructed that $BC = 2DE$. Now, the segment AG matches every point like F on DE with a unique point G on BC. But since BC is twice the length of DE, are there twice as many points in BC? "Yes, because the line is twice as long, and twice infinity is still infinity." And "No, because of the one-to-one correspondence clearly created by AG." What an enigma! I think Galileo shoved this diagram under the rug, or perhaps, into the fireplace.



And so it stood until priest and logician Bernard Bolzano (1781-1848) and mathematician Georg Cantor (1845-1918) set down the definition of actual infinity. They are of separate generations, Cantor picking up Bolzano's line of thought, which had almost been forgotten. They begin at the beginning, looking at a set of objects – any objects. How many of them are in the set? Well, we count them... but how is that done? Well, we match the objects one-to-one with the natural numbers $\{1, 2, 3, \cdots\}$. This is just about as basic as it gets (in fact, the natural numbers are sometimes called the counting numbers). Now, if the set of objects matches only up to some natural number n, then this set of n objects is called **finite**. But if the set matches one-to-one with the *entire* set of natural numbers, then the set is called **countably infinite.** We have lots of mathematical examples: the set of natural numbers itself, the set of positive unit fractions $\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ of Egyptian fame, and the set of primes $\{2, 3, 5, 7, \cdots\}$. These sets are actual infinities in quantity. They are not potentially infinite, for no process is being used to generate them. They are already before us in their entirety. To be sure, one could define a process that generates the unit fractions, for instance, but that isn't the point anymore.

Dipping back into the physical world, few, if any, sets of physical things are countably infinite. The set of particles in the visible universe is large, roughly 10^{80}, but finite. Perhaps the set of forces that your arm and a string direct when revolving a stone around your head is infinite, since at every moment the pull is in a different direction. But this assumes that space and time are continuous, not quantized. We are dealing with the philosophical question of what is knowable "out there." Thus, I'll let you ponder that over a cup of java (remember, these paths have been walked by profound thinkers).

But Galileo's awesome paradox about the triangles still doesn't submit to this definition of infinity. We need Bolzano's insight. He said that a set A is **infinite** (an actual infinity) precisely when one of its proper subsets can be matched one-to-one with the entire set. A proper subset of A is a part of A which is lacking at least one element of A. Let's clarify all that by applying the definition first to the familiar set $A = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$ above. The set $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right\}$ is a proper subset of A since it is missing all the odd-denominator elements. OK, now for the one-to-one matching requirement, shown in a neat table:

$$
\begin{array}{lccccccccc}
A: & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{101} & \frac{1}{102} & \cdots & \frac{1}{n} & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & & \updownarrow & \\
B: & \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{202} & \frac{1}{204} & \cdots & \frac{1}{2n} & \cdots
\end{array}
$$

Therefore, A is an actually infinite set according to Bolzano's new definition. This dovetails perfectly with the previous discovery that A is a countably infinite set (remember, math is cumulative). Watch how muddled thinking is cleared away. Although at first we think that A has twice as many elements as B, the one-to-one matching shows they are equally numerous! The error in this was that comparisons like "twice" apply only to finite sets, with which we are so comfortable. Next, no finite set whatsoever can be matched one-to-one with a proper subset of itself. Euclid had affirmed this when he stated that the part is less than the whole, excluding infinite quantities and lengths.

So far, then, the Bolzano-Cantor definition has segregated countably infinite sets from finite sets. Next, let's go after Galileo's paradox. In the figure above, place a number line on segment DE, with 0 at D and 1 at E (any two numbers would do just as nicely). Now translate DE down to BC so that D and B coincide, both being the location of 0. Then BC is covered by a number line from 0 to 2, since it is twice as long by construction. Two things happen: (1) DE becomes a proper subset of the points of BC, and (2) a point F in DE at distance x from D (which was 0) is matched one-to-one with point G in BC at distance $2x$ from B (which was also 0). Conclusion: the set of points constituting BC is infinite by the new definition, and is behaving according to its nature. Galileo's paradox vanishes. Notice that this time, countable infinity is not relevant.

No symbols for infinity have been introduced yet. This will bring in a closely related line of reasoning. When mathematicians, from the late 1700s on, began talking about a limit, it meant that a potential infinity was being proposed. Let's use once more the set $A = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$, but now, insist in keeping it in size order, for sets have no inherent order. This produces the *sequence* $A = \left\langle\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\rangle$. We can now say that the 453rd term is $\frac{1}{453}$, without ambiguity. As we run along this sequence, the limit of its terms is 0, although none of the terms is zero. This is written as $\lim_{n \to \infty} \frac{1}{n} = 0$. The sequencing process yields 0 by way of potential infinity. In the precise words of Augustin-Louis Cauchy (1789-1857), one of the five greatest mathematicians, "When the successive values attributed to a variable approach indefinitely to a fixed value, ... this last is called the limit of all the others." [From his *Cours d'analyse de l'Ecole royale polytechnique*, 1821, pg 4, available at https://gallica.bnf.fr/ark:/12148/btv1b8626657t/f9.image]

The symbols "$n \to \infty$" are properly read as "*n* increases without bound," but this often becomes the incorrect "as *n* increases to infinity," or "as *n* approaches infinity." These two phrases are impossible, for there is not point on the number line that locates infinity. This is a misuse of the ∞ symbol. Recall one of the replies to Galileo's paradox was that twice infinity is still infinity. In symbols, $2(\infty) = \infty$. The only real number that solves this is 0, and ∞ certainly is not 0. Thus, ∞ is not a real number, so using the axioms of real numbers with it can lead to crazy results. Here is another one: $\infty + 1 = \infty$. In a way, this is reasonable. But no real number at all satisfies the equation.

Georg Cantor took the next decisive step when he assigned a name and symbol to an actually infinite set. He was influenced by theology from both the Old and the New Testaments, so that the infinite was not just mathematical, but an intimation of God's infinite intellect. [See Joseph W. Dauben, *Georg Cantor His Mathematics and Philosophy of the Infinite*, Princeton U. Press, 1979, pp. 146-148 and 228-232.] He called infinity a **transfinite cardinal**, and its symbol was the first letter of the Hebrew alphabet, $\aleph$, "aleph." The **cardinality**, or size, of any countably infinite set, like the set of natural numbers, would be $\aleph_0$, "aleph null." So also, the cardinality of set *A* above is $\aleph_0$. Then some things became clear. For example, $\aleph_0 + 1 = \aleph_0$ meant that if we insert another number into *A*, its cardinality would remain $\aleph_0$. Indeed, if we add or subtract any *finite* number of elements from *A*, the cardinality doesn't change, so $\aleph_0 \pm n = \aleph_0$ is logical. In this light, the last stanza of "Amazing Grace" is stunning:

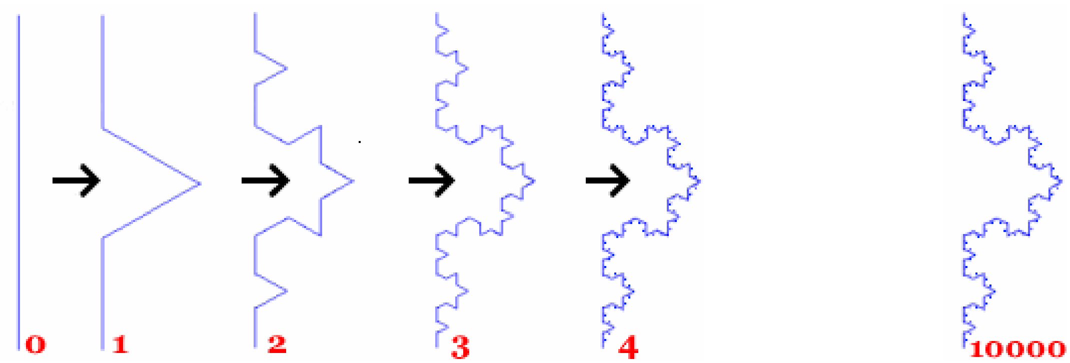
"When we've been there ten thousand years,
Bright shining as the sun,
We've no less days to sing God's praise,
Than when we first begun."

This describes an actual infinity of years with God as $\aleph_0 - 10000 = \aleph_0$. Perhaps Cantor sang this hymn; he would have heartily approved.

But Cantor went much further. He proved first that the set of all rational numbers (all fractions, which includes all the natural numbers, if you recall a past column), had the same cardinality as the set of naturals. This flies in the face of common sense, doesn't it? But the rationals actually can be put into a one-to-one match with the natural numbers, so the cardinality of the rationals is $\aleph_0$. The status quo in mathematical circles was already disturbed by Cantor's publications in the 1880's, and he was vilified by some of his colleagues. But he continued his study of infinite sets. Then, in 1891, he discovered a theorem so astonishing that the mathematical world split in two: those who accepted the theorem, and those who didn't. It is this: the set of real numbers has a cardinality that is far beyond even $\aleph_0$. The set of reals as an entity is so "huge" that its cardinality makes $\aleph_0$ seem like nothing! Cantor named this cardinality *c*, for "continuum," since the real number line was the continuum of the reals (remember our one-dimensional continuum in the last column). The theorem was symbolized as $c > \aleph_0$.

The proof of the theorem was just as novel as the theorem itself, and became known as Cantor's diagonalization. The gist of it isn't hard, so I leave it to you to read it. The result is that the cardinality of the reals is so immense that they cannot be counted. In other words, trying to match them one-to-one with the natural numbers inevitably leaves infinitely many reals uncounted. Thus, sets with cardinality *c* are called **uncountably infinite**. This term is not flimsy poetic license. Such sets literally cannot be counted in any way. Cardinality became important in the 20th century in order to study the ever more complicated functions that the sciences and math were demanding. And in the 1930's the diagonalization idea was used by logician Kurt Gödel (1906-1978) to show that there will always exist statements whose truth is undecidable.

Below are the first five steps in constructing the Koch curve. Starting at step 0, a plain line, bump up the middle third of the line into a small equilateral triangle (minus its base). This gives us four smaller segments in step 1. In step 2, each of those segments has had its middle third bumped up into a smaller equilateral triangle (minus its base), so that there are five peaks and 16 smaller segments. This bumping up procedure continues in steps 3 and 4. At step 4, the screen resolution can barely discern the smallest triangles. But we have just begun. Step 10,000 is conceivable, but not visible in toto. However, we can magnify a small part of it. You see an electron microscope magnification of a tiny piece of step 10,000 below, at right. It is expensive to set up the microscope and rent time on it, but I'm kidding. Marvelous as the machine is, it is unnecessary, for the image will look like step 4 (I copied and pasted it). We are observing the effect of self-similarity at *all* magnifications! Now imagine, if you can, the limiting curve at "step ∞." This would be the Koch curve, a fractal. We can now say that it represents an actual infinity of steps.



Luis F. Moreno, professor emeritus, Mathematics