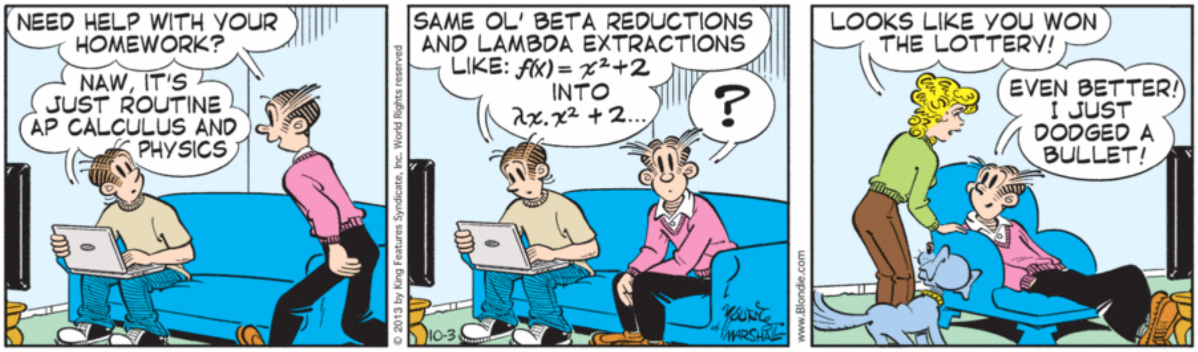

Math to Ponder While the Wave Passes

In the last column, we studied infinity from a technical, mathematical viewpoint. now, I'd like to write about mathematics from the view of the amazing language it is, unlike any other.

By language, here, I focus on the written form of communication that carries certain rules. This is intended to exclude spoken language, art, symbols with ad hoc meaning (such as codes), ritual dance, etc. A (written) language has an **alphabet**, or set of symbols it uses. If we include all the symbols in a typical keyboard, I find 95 characters, counting the space—the most common symbol of all. (Aha! "Nothing" is really important in this case.) A student taking a history course is as happy as a clam writing term papers using a word processor with only this set of symbols (well, maybe the happiness allusion is a bit rosy). With access to perhaps a dozen accent marks, the Romance languages come under the umbrella of the word processor. Beyond that, it is only necessary to scroll down the typographic collection under "Insert Character" in your processor to see the thousands of symbols needed to communicate worldwide.

But of course, something more is needed, or else the list "ew9qq&2bpt<M@P)aA" would convey meaning. Thus, along with an alphabet, we need a **syntax**, which is a set of rules which define how the symbols may be arranged and how they relate to each other. This isn't a grammar, but rather instructions on how to write meaningful lists of alphabetic characters, which we may call **well-formed formulas** (WFFs). Even so, some WFFs are far from clear, for instance, "Time flies like an arrow; fruit flies like a banana." Thus, meaning is elusive. We will stay with meaningful WFFs, and not try to define what "meaningful" means (this is the job of semantics).

In the common languages, a paragraph is a related set of sentences. It often conveys more meaning than the sentences separately. A sentence is a connected set of words. Likewise, the sentence conveys more meaning than its separate words. A word is dissected into its letters, and the word conveys meaning, while its constituent letters do not. We can stop the dissection here, the written letter is the indivisible atom for our purposes. Let's call this the "language hierarchy."

Next, most languages are linear, i.e., they are written and read in one direction, as you are doing here. They are one-dimensional. Hebrew is written and read from right to left, Chinese from top to bottom, and some ancient Greek was written like this:

> If four numbers are proportional, then the number produced from the first and fourth
> equals the number produced from the second and third; and, if the number produced
> from the first and fourth equals that produced from the second and third, then the four
> numbers are proportional.

So, be glad that your textbooks weren't written in this **boustrophedon** format, literally, "as the ox turns" in plowing. But all of these examples are still one-dimensional.

An example of a two-dimensional language is here:

The 9th Symphony,
Opus 125, mvt. IV,
Ludwig van Beethoven

This is read left-to-right, but additionally, the vertical position of notes is crucial. To be fair, accents in common language are located in the 2nd dimension, above and below letters, but they are far less critical to a composition than are the vertical position of notes in, say, the 9th Symphony. Anyway, an accented letter may simply be considered as another one-dimensional symbol in the alphabet, as in "n" and "ñ" in Spanish.

All the above is a prelude to the mathematical language. The selection from the Greek is actually Proposition 19 from Book VII of Euclid's *Elements*, a compendium which we have mentioned often in past columns (thankfully, the original was not written in boustrophedon) [from https://mathcs.clarku.edu/~djoyce/java/elements/bookVII/bookVII.html]. Up to the mid-1500s, practically all mathematics was written longhand, which was a strain on the reader as well as on the very fibers of the language. By the 1630's, many standardized symbols had appeared, and there must have been a realization of the power of abstract expression. Were you able to understand what Proposition 19 meant? Courtesy of Western mathematics, here it is: $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. A great deal happens in this small step, as we will see! One note must be added here. For Euclid, $\frac{a}{b} = \frac{c}{d}$ was numerically equivalent to $ad = bc$ since he did not recognize zero or negative numbers. Thus, today we would modify this to say, "Provided that neither b nor d is zero, $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$."

The alphabet of mathematics contains the numerical digits 0 through 9, parentheses, brackets, symbols for the four basic arithmetic operations +, –, •, / (and their variants), and several more operation symbols, such as $\sqrt{}$, which increase in number as students learn more math. Also, there are alphabetic letters as symbols for variables, constants, parameters, and so on, taken from Latin, Greek, and occasionally gothic, Russian, and Hebrew. There are also several two- or three- letter symbols, such as dx and cos, which again come into focus as more math is learned.

Mathematical syntax is flexible, but conveys precise meaning. Thus, "It was the best of times; it was the worst of times,..." has an infinite variegation of meaning (as Dickens intended), but $(a+b)^2 = a^2 + 2ab + b^2$ is very precise in meaning and implication. This and $\frac{a}{b} = \frac{c}{d}$ show that mathematics is not linear, but two-dimensional, that is, one must not confine the eye to one line of symbols. More obvious in its two-dimensionality is the expression called a matrix, which is a rectangular array of quantities, for example,

$$\begin{bmatrix} 1 & a & b & c & d & 6 \\ 2 & e & f & g & h & 5 \\ 3 & \pi & j & k & \ell & 4 \end{bmatrix}.$$

A matrix may seem a bit artificial, although we can see a passing similarity to the flowing green characters in "The Matrix." But here is a two-dimensional expression that illustrates the fascinating capacity for math symbols to compose with themselves. Consider the fractional expression $1 + \frac{1}{a}$. But what if a itself involves a fraction, as $a = 1 + \frac{1}{b}$. Then we have $1 + \cfrac{1}{1 + \cfrac{1}{b}}$. Now, suppose that b is actually $1 + \frac{1}{c}$. Then the expression becomes $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{c}}}$. But why stop there? c could equal $1 + \frac{1}{d}$, and so on without end. The fraction would appear as $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$. This dizzying possibility is called a **continued fraction**, and they have great mathematical interest. This particular one can be evaluated, even though the process is endless (see the last column, about potential infinity). To evaluate it, write $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$, and observe that x reappears as the denominator of the fraction on the right. Thus, $x = 1 + \frac{1}{x}$, which has two solutions, the positive one being $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. This happens to be one of the most famous constants in the ancient history of math, called the Golden Ratio, or the Golden Section. Architects and artists loved it.

There are expressions that require us to think in more than two dimensions. For instance, deeply embedded in a proof from statistics, one finds this little monster: $\sum_{k=1}^{n} \sum_{l \neq k} \sum_{s \neq l,k} A_{kl} A_{ks} X_k^2 X_l X_s$ [from *Probability and Mathematical Statistics*, by Z. W. Birnbaum, Harper, 1962, p. 171]. The presence of the three variables k, l, and s indicates that a three-dimensional array of quantities is being added, in a certain order. Thus, mathematics is a **multi-dimensional** language. By the way, to write that summation in plain English would make a sentence as convoluted as a bowl of spaghetti, and the meaning would be just as hard to tease out as would one strand of pasta.

Let's return to $(a+b)^2 = a^2 + 2ab + b^2$. This is a formula for expanding the binomial $a + b$, when read left-to right. I tell students to get used to reading it *right-to-left* also, whereupon it becomes a way to factor the trinomial $a^2 + 2ab + b^2$. In common language, this would be like reading "...,times of worst the was it ;times of best the was It" and expect the reader to extract an unexpected, novel, interpretation! Obviously, no common language can do this. But math does this regularly. Thus, math is **bidirectional**.

There is another way in which bidirectionality often appears. It is derived from the underlying engine that drives mathematics: logic. Proposition 19 above has the curious phrase "if and only if." This is called a **biconditional** in logic, so that Proposition 19 is a biconditional theorem (of course, with the provision that neither b nor d is zero). Dipping just a little into the classical logic discovered by Aristotle, an **implication** or conditional statement is a sentence of the form "If A then B," where A and B represent any statements that can be either true or false (no grey areas allowed). Suppose that the implication itself, as a whole statement, is true. For example, "If I drive five miles with my old car, then it had gas in the tank," is a true implication, with A = "I drive five miles with my old car," and B = "it had gas in the tank." Being a true implication, we can then affirm that whenever A is itself true (I *did* in fact drive the five miles), it will follow, as the night the day, that B must also be true—logicians love to say "necessarily true." However, in the case that A is false, then nothing at all can be affirmed about B. That is, B may be true or false, so we are totally in the dark about B, even though the implication as a whole is true. Thus, if I *didn't* drive five miles, then we can't say much about the old car having or not having gas. It is interesting that legal language sometimes takes pains to assert biconditionality.

Now, Proposition 19 is an implication which Euclid proved; thus, it is true. It says, in part, "If $ad = bc$ (this is statement A, taken as given and true for four numbers we select), then $\frac{a}{b} = \frac{c}{d}$ (this is statement B, necessarily true)." So far, so good. But more than that, it says, "only if," which is shorthand for the converse statement "If B then A." With this, we can affirm that whenever B is true, then A follows as true. So, we have here a two-way street in logic. If we start with true $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$ is true, and *furthermore*, if we start with a true proportion $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ is true. This two-way implication is what biconditionality means, and in math and logic, it means that the two statements are **equivalent**. All biconditional theorems are bidirectional statements in logic.

Now, let's consider the language hierarchy of mathematics. It wouldn't be objectionable to roughly equate a paragraph in written prose with a mathematical theorem. I choose the detailed and important Fundamental Theorem of Integral Calculus (Part I), which says, "Let f be a Riemann integrable function over $[a,b]$ and let F be an antiderivative of f over $[a,b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$." The syntax is precise, and it is an implication. We are not interested in what it says, but in its constituent parts. The words themselves don't have much connotation, meaning that we can't develop fuzzy senses about words like "antiderivative" and "Riemann integrable." On the other hand, "Riemann integrable" has an enormous denotation, or direct meaning, which can easily require a full chapter in some books. So, already, mathematical terms have a larger set of direct implications than most common words, and most of that structure will be essential in understanding the theorem. We have said that the hierarchy of the words ends at their letters. But not so with the symbols. For instance, $[a,b]$ is a concise, precise abbreviation for a closed interval of the real number line, from a to b, and "closed" has, in itself, a deep set of implications. The symbols $\int_a^b f(x)dx$ are crammed with another group of implications, being the symbol for the just mentioned Riemann integrable function. The hierarchy here involves considering the meaning of $\int_a^b \ldots dx$ and how it relates to $f(x)$, as well as to the constants a and b. The constituent symbol dx is not itself terminal, but carries many mathematical implications, which themselves have implications.

A good analogy is that the internal structure of anything like $\int_a^b f(x)dx$ is like a hypertext page on the Web. One could click on any of its constituent symbols, and be redirected to an entire page of information. In turn, that page will have numerous links to more meanings. All this is to say that a mathematical paragraph such as the Fundamental Theorem has a profound, linked hierarchy, far deeper than a standard paragraph of prose. This is what makes math abstract, and an increasingly powerful form of communication. Mathematics was the first hypertext language ever invented.

The increasing difficulty that all of us experience as we study more and more mathematics strikes me as analogous to being introduced to a video game. At the beginning, it has a few characters imbued with a few powers. The playing field is limited. It's a pretty easy game, as is arithmetic. But in time, you find a few dozen new characters, each with several powers at their disposal. And the arena you play in is large. This is like moving into algebra and trigonometry. After awhile, a hundred new characters enter, each with many powers to interact with and control. Now, the field is the size of a continent. The student is now grappling with calculus. Beyond that, hundreds of new characters may come in, some with specialized powers, others as avatars. And the field is now a planet, or even a galaxy. One is now a graduate student in a particular field of math. Unlike an actual game, no character ever dies in math. The best players, like Archimedes, Newton, Euler, Gauss, and Cauchy, are so imaginative that they modify the game and expand it. The only fundamental rule of the game is that it be internally consistent, by logic. The game is cumulative, and conquests are permanent, which has been a theme throughout these columns.

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