Math to Ponder While the Wave Passes

The two problems from the last column involving complex numbers are answered at the end.

At this point, we have made a grand tour of the famous types of numbers that mathematics supplies to civilizations as they advance. These were, in order of appearance, the whole numbers, the integers, the rationals, the irrationals, the reals, and finally, the complex numbers. We have covered a lot of territory.

Today, I open with the subtitle from a recent article, “I’m trapped. I feel two-dimensional...” [Paul Ford, in *Wired*, June 2020, pg. 17]. Common language is sprinkled with references to dimension. Informally, people know what two and three dimensions are, and it’s very sci-fi to talk about four dimensions. But what are these things, precisely? Mathematics is the place to find the answers, without any of the mumbo-jumbo often seen on the Web. Intuitively, a sheet of paper is two-dimensional, and a room is three-dimensional. But intuition is a fickle faculty. So, if the paper is rolled into a cylinder, then is it two- or three-dimensional? It is necessary to back up a little, and consider geometrically what a point and a line are.

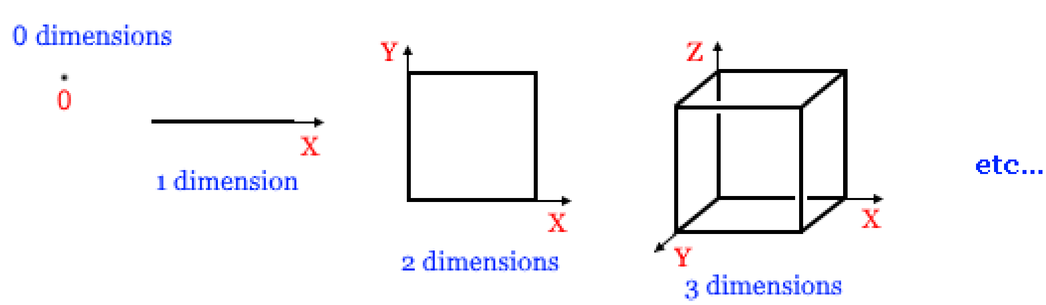
As Euclid stated in 300 BC, "A point is that which has no part." He meant that a point is not divisible into parts, which distinguishes it from a line segment. If all you have is a point, then it is your exact location. Nothing else is needed to specify your location, since literally there is nothing else around but that point. It will soon make sense to say that any single point has zero dimensions, or is **zero-dimensional**.

Next, take a line segment, which *is* divisible into parts (Euclid wisely steered clear of infinitely long lines). Let’s look at this process with care. On a line segment, label one endpoint as "0." Label the other endpoint "1." As Euclid would have it, this establishes a unit length which may be copied over and over, in both directions. The resulting segment can, for instance, stretch from -350 to $10,000$. (We pulled a fast one on Euclid, because he wouldn’t admit zero or negatives, but that is irrelevant to our investigation.) What about positions in between all the integers? These are located by all the rational and irrational numbers that have been the subject of past columns. We have promoted the featureless segment to what we call a "number line." The number line is now officially called a "continuum," because it has no gaps at all; every point has a real number label. Moreover, you need exactly one number to locate yourself on the line relative to 0. For example, -23.7 puts you at 23.7 units to the left of 0, and nowhere else. The fact that you need exactly one number to locate your position means that a line segment has one dimension, or is **one-dimensional**. It seems like we're splitting hairs, but the clarity is essential for what we are building.

Return to that sheet of paper. It is pretty clear that a single number line drawn on the paper is not enough to locate your position on it. You need two, crossed, number lines. This produces the Cartesian coordinate plane, with the famous X and Y axes, and the grid which we visualize as graph paper. The starting point, or origin, is labelled $(0,0)$, and your position on the plane is given uniquely by coordinates (x,y). This fact that you now need two separate real numbers to locate yourself means that the Cartesian plane has two dimensions, or is **two-dimensional**. Observe how this repairs intuitive thinking: rolling the paper into a cylinder doesn't change its dimension, since the grid is embedded in the page, and it still indicates that (x,y) is all you need. It is convenient (but not necessary) to draw the X and Y axes at right angles. We say that the coordinate axes are independent of each other. The distance from $(0,0)$ to any point (x,y) is given by the Pythagorean theorem as $d = \sqrt{x^2+y^2}$.

Things are now falling into place. Consider the room you may be in. Imagine a speck of dust floating in a sun beam. How do you locate it? Mathematically, we make the floor into a Cartesian plane, with X and Y axes along the lines where two walls meet the floor, and then draw a new axis vertically up from the corner, beginning at the origin, where the X and Y axes meet. The new axis is called Z. Now, the dust mote can be uniquely located as x units along X, y units along Y, and z units along Z. Its coordinates are (x,y,z), relative to the origin $(0,0,0)$. No fewer than three coordinates suffice to locate that speck, so the room has three dimensions, or is **three-dimensional**. Again, it is convenient to draw the three axes at right angles, especially since contractors are expert at building rooms that are square and plumb! The three axes are independent of each other, and the space of the room is also a continuum. By the way, the distance d from origin to mote is measured by the Pythagorean theorem, enhanced to three dimensions: $d = \sqrt{x^2+y^2+z^2}$.

One detail may wake you up in the middle of the night. In the case of the sheet of paper, we could draw three axes, X, Y, and Z, and then call the page three-dimensional, no? No, for we observe that any two of the three axes will locate your position, and the third axis will be unnecessary. Thus, return to sleep. This is related to the fact that no more than two axes may be drawn all at right angles to each other on the Cartesian plane. Likewise, in the room, a maximum of three axes can be drawn mutually at right angles, no more. The four situations are seen below.



We can now grasp the next step. A **four-dimensional** continuum would be a space in which four coordinates are needed to locate your position, as in (x,y,z,w). Since our space is three-dimensional, it isn't possible to visualize a space of four dimensions. But it can be handled perfectly well by mathematics. For instance, we could say that in four dimensions, four axes, X, Y, Z, and W, could be intersected, and each would be at right angles to all the others! Also, the distance d from the origin $(0,0,0,0)$ to your position (x,y,z,w) can be found. It is amazing that the Pythagorean theorem still holds. You will recognize it in its Superman outfit: $d = \sqrt{x^2 + y^2 + z^2 + w^2}$. We may conclude that space of four dimensions is something that is logically manageable, but not able to be visualized, even in our mind. Fortunately, a *projection* of a four-dimensional cube into three dimensions can be seen at https://en.wiktionary.org/wiki/hypercube

If the axes are not spatial, then more can be said. Albert Einstein (1879-1955) and Hermann Minkowski (1864-1909) showed in the special theory of relativity that three-dimensional distances would not be agreed upon by different observers. They discovered a four-dimensional world for events, but with a nuance. The fourth dimension would be time, which initially makes sense, for we can uniquely locate an event with three spatial dimensions, x, y, and z, and its time of occurrence, t. Thus, the four coordinates are (x, y, z, t). Judging by the lyrics, "I was in the right place, but it must have been the wrong time...," we're pretty sure that Dr. John knew about messing up the time coordinate. But time isn't like the other three dimensions, as even Dr. John would agree. Indeed, the Einstein-Minkowski Pythagorean theorem looks quite weird: $d = \sqrt{c^2t^2 - x^2 - y^2 - z^2}$, where c is the speed of light. This distance d is a combination of spatial and temporal parts, so Einstein called it a distance in the space-time continuum. We can't feel it, but every particle we are made of traverses its own space-time path. Their paths stick together as we move from room to room. If an eight-year-old loses a tooth, the space-time path of the child departs from the space-time path of the tooth. Nothing spooky is implied here. This is just a more universal view of reality than we are accustomed to.

If we go beyond physics, many things have four-dimensional qualities. For instance, a town's tax records may be accessed by year, town section, lot number, and name of owner. This effectively describes four coordinates, say, (1997, sec 22, lot 3-08, Smith). Mathematically, this is a four-dimensional space. As always, the coordinates are unique, for if one record applied to two events, the tax assessor would be in hot water. Beyond that, this verse from the letter of Paul to the Ephesians is arresting in this context: "that you ... may be able to comprehend with all the saints what is the width and length and depth and height—to know the love of Christ which passes knowledge." [Eph. 3:18,19]. Did Paul know that he was surpassing the geometry of Euclid in explaining the love of God?

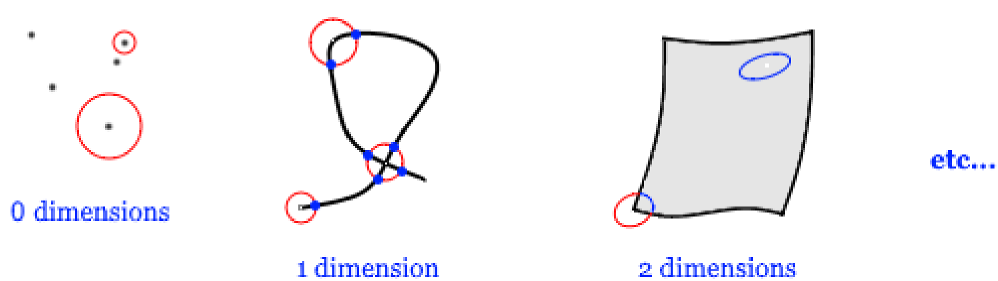
One cool idea that now makes sense is that our two-dimensional sheet of paper resides within the three-dimensional room. Officially, the paper is a two-dimensional **subspace**, which has many mathematical implications. In the same way, a line drawn on the paper is a one-dimensional subspace of the paper (and, of course, of the room). Thus, we know something about subspace communication, the technology that Vulcans invented to talk at faster-than-light speed with the Enterprise.

This idea of independent axes giving coordinates can naturally be extended to many dimensions. One version of string theory, a complex study of how all the particles and forces of the universe may be explained, works in 11 dimensions. This means that the harried physicists must pin down 11 coordinates in order to describe any event in the universe. This spins our heads, but wait 'till you hear what the quantum physicists require. They need an *infinite*-dimensional space! No problem at all... this is in fact called a Hilbert space, after the supremely ingenious mathematician David Hilbert (1862-1943). It has logical, understandable properties, since mathematics provides all the numbers, and spaces, that civilizations need. I think Mr. Spock would have loved to have met Hilbert.

It took some novel thinking to divorce the idea of dimensionality from the Cartesian coordinate mold, and yet not contradict it (which can never happen in mathematics). Notice that when we rolled the sheet of paper into a cylinder, we tacitly assumed that it was in embedded in three dimensions, which, strictly speaking, we had not defined yet.

It turns out that dimensionality can be defined without using coordinates. The absolute minimum that is needed is a collection of points that we can call a "point set," and some notion of "distance." Whatever we decide distance to be, it must follow these four requirements: (1) the distance between two points is zero or positive (never negative), (2) the distance from a point to itself is zero, and conversely, if the distance between point *A* and point *B* is zero, then *A* and *B* are the same point, (3) the distance from *A* to *B* is equal to the distance from *B* to *A*, and (4) the distance from *A* to *B* is less than or equal to the distances from *A* to *C* and *C* to *B* added, where *C* is any intermediate point. If the formula we use to measure distance satisfies these requirements, then it is called a **metric** (a precise name, highjacked by the news media so that they may sound sophisticated). Indeed, all the forms of the Pythagorean theorem seen above are metrics. But many other strange metrics exist, creating totally strange geometries. We won't use any particular metric in what follows, but know that it is being appealed to when we say "radius *r*."

Here we go. A point has dimension zero. This is because if we draw a circle with radius *r* around the point, the circle intersects the point zero times. Likewise any finite collection of points has dimension zero. See the figure below, left. Next, a line or a common curve has dimension one because when we draw a circle with radius *r* around any of its points, it always intersects the curve at points, which are dimension zero (below, center). Next, remember the sheet of paper, now just another point set. Around any one of its constituent points, draw a circle with radius *r*. It will always intersect the paper at a curve, namely, part of the circle (or the whole circle, below, right). The plane has dimension two because this intersection is of dimension one. The process goes on in this way. If you can draw a sphere of radius *r* around a point, then the space you are in intersects the sphere at the surface, which has dimension two. Therefore, the aforementioned space has dimension three. [See *Geometry and Intuition* by Hans Hahn, in "Space, Intuition and Geometry", reprint series RS-5 by the SMSG, 1967, pp. 36-38.]

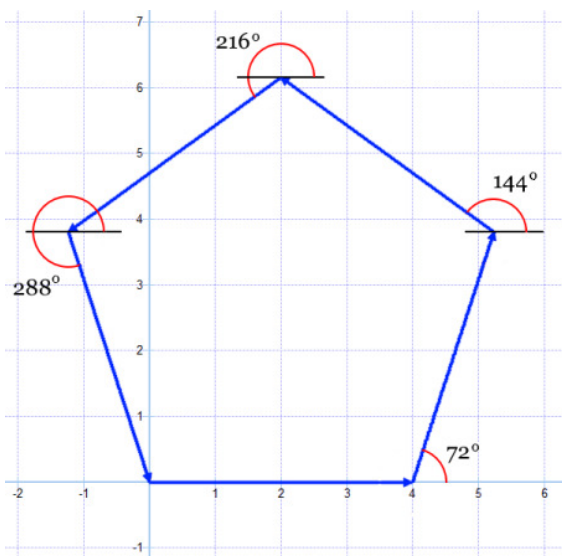


The process above is very discriminating in deciding what the dimension of a set of points is. But then, in the early decades of the 20th century, certain mathematical objects were constructed—we may call them limiting point sets—that were neither one-dimensional, nor two-dimensional. They were justifiably interdimensional, or **fractal**. Mathematicians Felix Hausdorff (1868-1942) and Abram Besicovitch (1891-1970) extended the definition of dimension so that it would cover these strange creations. As always, new definitions never contradict past ones, so our humble sheet of paper still has the Hausdorff-Besicovitch dimension two. In contrast, a famous fractal exists called the Koch curve. The Koch curve is the limit of a simple process, containing "in the end" infinitely many, infinitesimal triangular spikes. As such, its full splendor can only be imagined. However, one important property of any fractal is that any small portion of it looks like any other small portion. This is called self-similarity at all scales, which is quite apparent in the Koch curve. And its Hausdorff-Besicovitch dimension is $1.2691\ldots$! Your homework assignment is to find this fractal curve on the Web, and see how it is created.

Now, the practical-minded among you may say, "Fine, but this takes us back to the ghosts of ghosts from centuries ago. You have built another castle in the air." Surprisingly, not at all. In the early 1960's, mathematician and meteorologist Edward Lorenz (1917-2008) was studying math models that computers could use to predict the weather. To his consternation, he found that rounding off data would not produce output close to that given by high-precision data. Instead, the output was far from expected, and unpredictable. What evolved from this was a curve whose dimension was about 2.06 (depending on some parameters), the "Lorenz attactor." The upshot was that many things in nature operate chaotically, meaning that tiny changes in initial conditions yield great, unpredictable, changes in final states. Lorenz christened this the "butterfly effect." In time, many math models were found that described chaotic behavior, so chaos no longer is associated with crazy. Chaos regularly produced fractal objects. One can't get more practical than weather prediction, but lurking just out of sight live fractal, interdimensional, things. It is uncany that fractals were discovered first as purely mathematical objects, and not long after, they were pressed into the service of science.

Even without a computer, fractal curves abound. Consider a map of Great Britain, in particular, its coastline, which we draw as a wiggly curve on the map. We can measure the length of the coastline, approximately, by laying a flexible thread over the curve, and then measuring its length. We realize that a larger map will yield a better approximation of the length, and the larger the map, the better. Then, we discover that any small piece of the curve is essentially the same as any other piece (i.e., you would need a photo of a shoreline to determine if it was a part of Wales or of Scotland). This self-similarity is the calling card of a fractal curve. So, the coastline of any island (and hence, that of any lake) is ultimately a fractal! In Great Britain's case, the coastline was found to have dimension about 1.25 [all these dimensions are from the superb Wikipedia article "List of fractals by Hausdorff dimension"]. When I was a boy, I craved some comic books entitled "Tales to Astonish." Interdimensional objects are the real tales to astonish, I think.

The previous column asked you to add the complex numbers with angles 0°, 72°, 144°, 216°, and 288°, all with magnitude 4. Here they are in tip-to-tail fashion. Voila! A perfect, regular pentagon.



The second problem (a bit trickier) was to find the roots of $3x^2 + 4ix + 5 + 6i = 0$ by the quadratic formula. It says $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and in this problem, $a = 3$, $b = 4i$, and $c = 5 + 6i$. Substituting, we get $\dfrac{-4i \pm \sqrt{(4i)^2 - 4(3)(5+6i)}}{2(3)} = \dfrac{-4i \pm \sqrt{-76 - 72i}}{6}$. We can leave it like that, or simplify a bit to get $\dfrac{-2i \pm \sqrt{-19 - 18i}}{3}$. But are these two complex numbers actually roots? Substituting the + case into the given quadratic gives $3\left(\dfrac{-2i + \sqrt{-19 - 18i}}{3}\right)^2 + 4i\left(\dfrac{-2i + \sqrt{-19 - 18i}}{3}\right) + 5 + 6i$. Does this equal to zero? Several algebraic steps later, it in fact does become zero, as if by magic. The – case also checks out. Therefore, the quadratic formula seems to work even with complex numbers (it will *always* work).

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