"Find out who set up this experiment. It seems that half the patients were given a placebo, and the other half were given a different placebo."

Math to Ponder While the Wave Passes

The last essay left us with a solid foundation for real numbers. But long before those axioms were identified and collected, certain unwelcome phantasms began to appear, and they refused to remain in a bottle. Thinking as a 15th century mathematician might have, these quantities were something like the ghosts of ghosts. They are still called "imaginary" numbers, although they are not at all figments of the imagination.

Historians argue about whether the Renaissance began with the publication of the *Divine Comedy* in 1320. For our purposes, let's take that as a convenient demarcation from Medieval times. The Renaissance was driven by the desire to rediscover and interpret ancient civilization in humanist terms. I think that a second driving force existed. It was the desire to surpass the ancients, for it was assumed that they had possessed some kind of vital knowledge long lost to the West. In literature, the *Divine Comedy* did not surpass the *Aeneid*, but it certainly equalled it. In sculpture, the statue of David certainly equalled the skill of Greek sculptors. In architecture, it looked like things were also tied, with geometrical symmetry greatly prized. But in art, there had been a clear breakthrough! The development of perspective was something new, and better, than the ancients had done. And interestingly, it was a fresh application of good old Euclidean geometry. Also, astronomy had its breakthrough at the end of the hard-fought contest against the ancient *Almagest* of Ptolemy. (The idea that the Earth orbited the sun was revived by mathematician and cardinal Nicholas of Cusa in 1444, a century before Copernicus.) Again, astronomy used geometry as its base of operations. A third breakthrough came in the more arcane field of algebra, so it's not as well celebrated.

The mathematical world of Italy in the sixteenth century had something of the flavor of the Wild West in America. Instead of cowboys, independent mathematicians vied for survival and fame. Rather than gun duels, they ran problem contests to see who was the best at solving them. In 1490, Luca Pacioli, one of the best mathematicians in Europe—and Leonardo daVinci's personal tutor—gave clear notice that a desperado was out and about. Neither the Greeks nor the Arabs after them had been able to solve the general cubic equation, which for the record is $ax^3 + bx^2 + cx + d = 0$ (a, b, c, and d are given numbers). And there was a bounty on this thing's head: fame and financial security would come to whomever could capture the solution. Life was tough for mathematicians back then. If you were a mediocre one, you might earn a living by making horoscopes for gullible aristocrats. If you were a good one, you could attach yourself to the court of one or another ruling family, or attract a famous patron (like daVinci). Moreover, you kept your discoveries close to the vest, or else your competitors would get the advantage! [The next five paragraphs use Chapter 9 of *A History of Mathematics, an Introduction*, by Victor J. Katz, Harper Collins, 1993.]

In the early 1500's, an obscure mathematician named Scipione del Ferro stitched together what was known about the cubic, and began posing problems about it that no one could solve. Of course, he kept his new method secret, and upon his death, only a student of his knew the method, but poorly. A brilliant fellow named Niccolo Tartaglia (1499-1557) eventually found the method and was able to expand it into a general algorithm. The ultimate showdown occurred when Tartaglia challenged the top mathematician of the time, Gerolamo Cardano (1501-1576), who was also something of a gambler and swindler. Cardano was not able to solve the cubics that Tartaglia fired at him, and more importantly, Tartaglia did solve whatever cubic Cardano posed. Thus, Tartaglia's name rose like a rocket.

It happened that Cardano was writing one of those compendiums of all the mathematics known at the time (Pacioli had written one in his days). He knew that it would fall short of the mark if the full solution to the cubic were missing. Thus, he cajoled Tartaglia into revealing the algorithm, with a solemn promise never to divulge it. And so it was that Cardano's *Ars Magna* was published in 1545, complete with Tartaglia's algorithm! Was it any better that Tartaglia was acknowledged as the discoverer? I think not, for Tartaglia lost fame and fortune and died almost unknown.

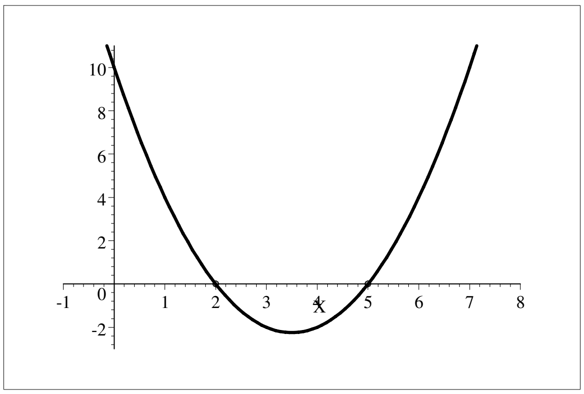
Well, despite an atmosphere resembling *The Good, the Bad, and the Ugly*, Renaissance mathematics had surpassed classical math in a theoretical, substantial way. And the engine of discovery would accelerate, because between the time of Cardano's book and 1650, much of the algebraic symbolism that we recognize today was established. Never again would algebra be hobbled by horrendously long sentences.

The cubic equation had been conquered. Or had it? Sometimes, depending on the values of a, b, c, and d, the square root of a negative quantity would unavoidably crop up. One could obtain $\sqrt{-16}$, for instance. Cardano dismissed such answers as unusable, for no number squared could give −16. Remember from a previous column that in those times, negative numbers were considered a little ghostly, the output of some algorithms, but to be cast off whenever possible. Hence, the square root of a negative must have seemed like the ghost of a ghost, as I called it above. The cubic equation hadn't been solved in a fully understandable way after all, and no one knew what to do with that. Nevertheless, mathematicians have a nose for the uncanny.

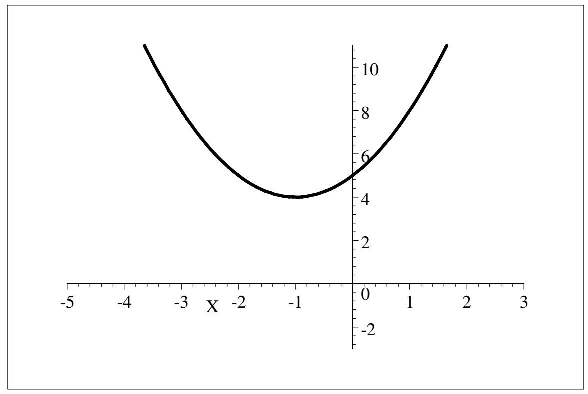
It was Rafael Bombelli (1526-1572) who first faced off against imaginary numbers. (A most mellifluous name; if I owned a spacecraft, I would christen it with his name.) He identified the **imaginary unit** $+\sqrt{-1}$ as the source of all the difficulties, calling it *più di meno* (plus of minus), and just to be ultra clear, $-\sqrt{-1}$ was *meno di meno* (minus of minus). This allowed him to work with solutions to the cubic involving $2+\sqrt{-121}$ and $2-\sqrt{-121}$ as if they were actual numbers, not anomalies to throw away. He then proceeded to discover the arithmetic of imaginary numbers, including the fundamental idea that $a\sqrt{-1} \times b\sqrt{-1} = -ab$. It was pretty clear that imaginaries did not mix at all with reals in addition. Thus, a **complex number** was the indicated sum of a real and an imaginary, as in $2+\sqrt{-121} = 2+11\sqrt{-1}$. Today, we would write this as $2+11i$, where i is a more convenient symbol for $+\sqrt{-1}$.

It is curious how the arithmetic of complex numbers was worked out with the difficult cubic equation, and then applied to the much easier quadratic equation, $ax^2+bx+c=0$. Recall that the solution to this was known thousands of years before, and we write it as $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, a formula which no doubt brings a tear to your eye (of joy, I hope, not of frustration). The solutions were called the "roots" of the quadratic. Notice the radical. If $a = 1$, $b = 2$, and $c = 5$, we get $\sqrt{4-20} = \sqrt{-16} = 4i$, one of Bombelli's *più di meno* imaginaries. In today's algebra, we would write the two roots of $1x^2 + 2x + 5 = 0$ as $x = \frac{-2 \pm 4i}{2} = -1 \pm 2i$.

When Descarte's graphs of equations came along in 1637, some of the mystery of complex numbers was removed. For the first time, a parabola of Greek fame could be seen as the curve given by $ax^2 + bx + c$. The roots of a quadratic could also be seen as the points where the curve crossed the X axis, as for the parabola at left, below. However, the parabola in the previous paragraph is on the right, and it obviously has no X intercepts. Where, then, are the roots given by the quadratic formula? They are complex, as we have just seen, namely $-1 + 2i$ and $-1 - 2i$. Aha! Complex numbers were invisible in Cartesian coordinates: invisible, but logical!

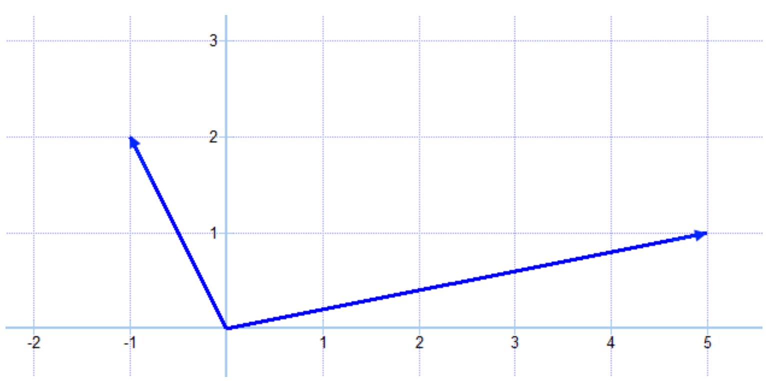


$1x^2 - 7x + 10$



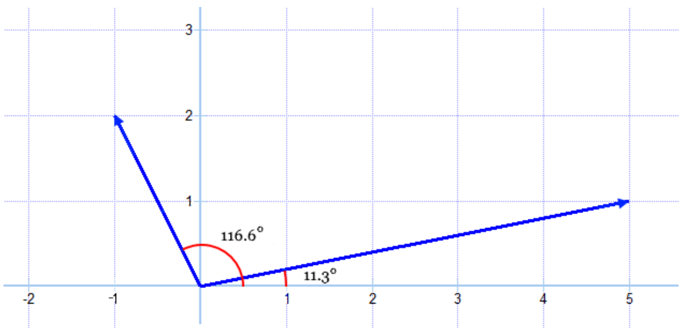
$1x^2 + 2x + 5$

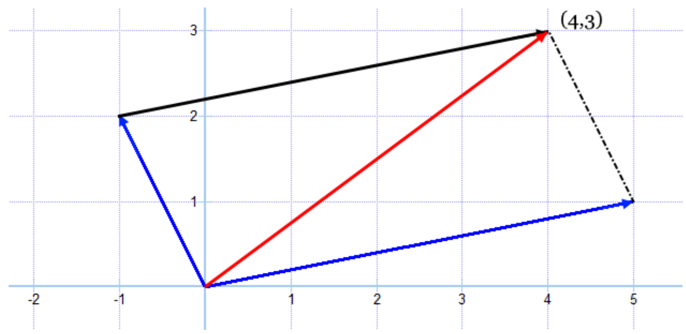
We move forward to the time of Leonhard Euler (1707-1783), who, along with Archimedes and Newton, is one of the five greatest mathematicians of all time. Within the massive amount of new mathematics produced by Euler (pronounced oi'ler, as in the German), we find a new viewpoint for complex numbers. It starts off almost trivially. He writes the complex number $-1 + 2i$ as the point $(-1, 2)$. We have just learned that complex numbers are invisible in Cartesian coordinates, but Euler tells us not to use real numbers on both X and Y axes, but only on the X axis. The Y axis will now harbor all imaginary numbers, like $-3i, -2i, i, 2i, 3i, \cdots$. The only real number which they both contain is zero, because after all, $0 = 0 + 0i$, which matches the origin point $(0, 0)$. And suddenly, $(-1, 2)$ is visible, not a ghost anymore. Next, Euler says that we should think of $(-1, 2)$ as the arrow, or **vector**, from the origin to $(-1, 2)$. You see it below along with one of its brothers, $(5, 1)$. Thus, complex numbers are arrows on graph paper.



With this viewpoint, addition and multiplication become really interesting. The addition $-1 + 2i + 5 + i = 4 + 3i$ was known to Bombelli. In vector format, this looks like $(-1, 2) + (5, 1) = (4, 3)$. Nothing up Euler's sleeve... yet. But now, he pulls out an ace, for vectors were the way to analyze forces, known since the days of Galileo. And Euler finds that complex numbers add like forces add. Let's see how they do that.

The strength or magnitude of a force is given by the length of the arrow, which comes from the Pythagorean theorem. So the magnitude of the force vector $(-1,2)$ is $\sqrt{(-1)^2+2^2} = \sqrt{5} \approx 2.236$ pounds (forces are measured in pounds, or newtons in the metric system). The longer vector above has magnitude $\sqrt{1^2+5^2} = \sqrt{26} \approx 5.099$ pounds. We are familiar with this idea, because magnitude of force = strength of pull. Now, the arrows point in different directions. These each correspond to the directions of pull. By tradition in math, the direction of a vector is given by the angle from the positive X axis. As seen below, $(-1,2)$ is at about $116.6°$, and $(5,1)$ is at about $11.3°$ (if you have studied trigonometry, you should know how to find those angles).





The addition of forces is done by moving one vector in parallel so that its tail is at the other's tip. This is seen above, right. The sum of the blue vectors is the red vector, which points to $(4,3)$. This red force is in the direction that a bowling ball would move when pulled by the blue forces acting at the same time. The figure above right is what Galileo, physics, and engineering, call the *parallelogram of forces*. This matches exactly with complex number addition, just as we found above, $(-1,2)+(5,1) = (4,3)$. So, complex numbers add as forces do; they are not imaginary at all...

However, forces don't multiply, while complex numbers do. Euler finds out how to visualize that, and it's amazing. Two complex number vectors multiply by multiplying their magnitudes, and adding their angles. Thus, $(-1,2)\times(5,1)$ comes out with magnitude $\sqrt{5}\times\sqrt{26} \approx 11.4$, and in direction $116.6° + 11.3° = 127.9°$. We see it below. On the other hand, Bombelli would have done it as $(-1+2i)(5+1i) = -7+9i$ (skip the details if you wish). Do Euler and Bombelli agree? Well, the magnitude of $-7+9i$ is $\sqrt{(-7)^2+9^2} = \sqrt{130} \approx 11.4$, and the angle of the vector $(-7,9)$ is $127.9°$. Absolutely perfect! A theme of these columns is that math is cumulative and permanent. A lot of mathematics was discovered in the 200 years between Bombelli and Euler, and it was all cumulative and permanent.

Are these curiosities used today? Yes, in many fields. Here is Richard Feynman, one of the great physicists of the 20th century (and Nobel winner), explaining some math behind quantum electrodynamics. “Since these arrows obey the same rules of algebra as regular numbers, mathematicians call them numbers. But to distinguish them from ordinary numbers, they’re called "complex numbers." When an event can happen in alternative ways, you add the complex numbers; when it can happen only as a succession of steps, you multiply the complex numbers.” [*QED The Strange Theory of Light and Matter*, by Richard P. Feynman, Princeton Press, 1985, pg. 63.]

One last fact about complex numbers, $a + bi$, where a and b are real. They follow all the axioms for the real numbers listed in the previous column, except for one: axiom (9), which distinguishes positives from negatives. Thus, there is no such thing as a positive or negative complex number, when $b \neq 0$. An unexpected consequence of this is that inequalities don’t exist between them. So, anything like $14 + 100i > 2 + 7i$ is meaningless (to be sure, the *magnitude* of $14 + 100i$ is much larger than that of $2 + 4i$).

I leave you with two nice problems with complex numbers. For the first one, here are some multiples of 72°: 0°, 72°, 144°, 216°, 288°. On a graph paper, start at the origin, and draw the vector for the complex number with magnitude 4 and angle 0°. From the tip of this vector, use tip-to-tail addition to add the vector with magnitude 4 and angle 72° (use a protractor!). To the sum (the tip of the second vector), add the vector with magnitude 4 and angle 144°. Keep adding vectors with magnitude 4 and the listed degrees. What have you made?

The second one is more challenging. Remember that the quadratic formula (it’s above) is routinely used to solve equations like $3x^2 - 2x - 1 = 0$. Does it work on $3x^2 + 4ix + 5 + 6i = 0$? Try it and check your solutions.